General Instructions:

(i) **All** questions are compulsory.

(ii) This question paper contains **29** questions divided into four sections A, B, C and D. Section **A** comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only **one** of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask logarithmic tables, if required.

Question 1

Find the acute angle between the planes $\vec{r}\cdot\left(\hat{i}-2\hat{j}-2\hat{k}\right)=1$ and $\vec{r}\cdot\left(3\hat{i}-6\hat{j}+2\hat{k}\right)=0$.

OR

Find the length of the intercept, cut off by the plane $2x+y-z=5$ on the x-axis

Solution

The vector equation of the planes is $\vec{r}\cdot\left(\hat{i}-2\hat{j}-2\hat{k}\right)=1$ and $\vec{r}\cdot\left(3\hat{i}-6\hat{j}+2\hat{k}\right)=0.$

It is known that if $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$, then the angle between them, is given by,

$$\cos\theta=\left|\frac{\vec{n_1}\cdot\vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|}\right|$$

So, the angle between the given planes will be

$$\cos\theta=\left|\frac{\left(\hat{i}-2\hat{j}-2\hat{k}\right)\cdot\left(3\hat{i}-6\hat{j}+2\hat{k}\right)}{\left(\sqrt{1^2+(-2)^2+(-2)^2}\right)\left(\sqrt{3^2+(-6)^2+(2)^2}\right)}\right|$$

$$=\left|\frac{3+12-4}{3\times7}\right|$$

$$=\left|\frac{11}{21}\right|$$

$$\Rightarrow\theta=\cos^{-1}\left|\frac{11}{21}\right|$$

OR

The given plane is $2x + y - z = 5$.

Dividing both sides of equation by 5, we obtain

$$\frac{2}{5}x + \frac{y}{5} - \frac{z}{5} = 1$$
$$\Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1$$

It is known that the equation of a plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where a, b, c are the intercepts cut off by the plane at x, y, and z axes respectively.

Therefore, for the given equation, the intercept made with the x-axis is $\frac{5}{2}$.

Question 2

If $y = \log(\cos e^x)$ then find $\frac{dy}{dx}$.

Solution

Let $y = \log(\cos e^x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\cos e^x)\right]$$
$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}(\cos e^x)$$
$$= \frac{1}{\cos e^x} \cdot (-\sin e^x) \cdot \frac{d}{dx}(e^x)$$
$$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$$
$$= -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{N}$$

Question 3

A is a square matrix with $|A| = 4$. then find the value of $|A. (\text{adj } A)|$.

Solution

We know $|A \text{ adj } A| = |A|^n$.
$|A| = 4$
$|A(adjA)| = 4^n$ (where n is the order of matrix A)

Question 4

Form the differential equation representing the family of curves $y = A \sin x$, by eliminating the arbitrary constant A.

Solution

Given $y = A\sin x$ $\quad\ldots\ldots(1)$
Differentiating with respect to x
$\frac{dy}{dx} = A\cos x$ .....(2)
From (1) and (2) we have
$\frac{dy}{dx} = \frac{y}{\sin x} \cdot \cos x$
$\Rightarrow \frac{dy}{dx} - (\cot x)y = 0$
Thus, this is the required differential equation.

Question 5

Find:
$\int x.\ \tan^{-1} x\, dx$

OR

Find:
$\int \frac{dx}{\sqrt{5-4x-2x^2}}$

Solution

Let $I = \int x \tan^{-1} x \, dx$

Taking $\tan^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x \, dx \right\} dx$$

$$= \tan^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left(x - \tan^{-1} x \right) + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

OR

$$\int \frac{dx}{\sqrt{5-4x-2x^2}}$$

$$= \int \frac{dx}{\sqrt{2\left[\frac{5}{2}-2x-x^2\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2}-(x^2+2x)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2}-(x^2+2x+1-1)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2}-(x+1)^2+1}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2}-(x+1)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2-(x+1)^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{(x+1)\sqrt{2}}{\sqrt{7}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}}\,(x+1)\right) + C$$

Question 6

Solve the following differential equation :

$\frac{dy}{dx}+y=\cos\ x-\sin\ x$

Solution

Given: $\frac{dy}{dx}+y=\cos x-\sin x$ .....(1)

This differential equation is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$

$P=1, Q=\cos x-\sin x$

$I.F.=e^{\int P dx}=e^{\int 1 dx}=e^{x}$

Now multiply (1) with the I.F. we get

$e^{x}\left(\frac{dy}{dx}+y\right)=e^{x}(\cos x-\sin x)$

Integrating both sides with respect to x.

$ye^{x}=\int e^{x}(\cos x-\sin x)dx+C$

$\Rightarrow ye^{x}=\int e^{x}\cos x dx-\int e^{x}\sin x dx+C$

$\Rightarrow ye^{x}=e^{x}\cos x-\int(-\sin x)e^{x}dx-\int e^{x}\sin x dx+C$

$\Rightarrow ye^{x}=e^{x}\cos x+\int e^{x}\sin x dx-\int e^{x}\sin x dx+C$

$\Rightarrow ye^{x}=e^{x}\cos x+C$

Thus, $ye^{x}=e^{x}\cos x+C$ is the required solution of the given differential equation.

Question 7

Find :

$$\int_{-\frac{\pi}{4}}^{0}\frac{1+\tan\ x}{1-\tan\ x}dx$$

Solution

Let

$$I=\int_{-\frac{\pi}{4}}^{0}\frac{1+\tan x}{1-\tan x}dx$$

$$I=\int_{-\frac{\pi}{4}}^{0}\frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}dx$$

$$=\int_{-\frac{\pi}{4}}^{0}\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Let

$\cos x-\sin x=t$

$-(\sin x+\cos x)dx=dt$

For $x=\frac{-\pi}{4},\ t=\sqrt{2}$ and $x=0,\ t=1$

$$I=-\int_{\sqrt{2}}^{1}\frac{dt}{t}$$

$$=\int_{1}^{\sqrt{2}}\frac{dt}{t}$$

$$=[\ln t]_{1}^{\sqrt{2}}$$

$$=\ln\sqrt{2}$$

Question 8

Let * be an operation defined as * : **R** × **R** ⟶ **R**, $a * b = 2a + b$, $a, b \in$ **R**. Check if * is a binary operation. If yes, find if it is associative too.

Solution

Let $a,b\in R$. Then,

$a+b\in R$ (Addition is a binary operation on R)

$\Rightarrow a+(a+b)\in R$ (Addition is a binary operation on R)

$\Rightarrow 2a+b\in R$

Thus, $a*b\in R$ for all $a,b\in R$.

Hence, * is a binary operation on R.

Let $a,b,c\in R$

$(a*b)*c=(2a+b)*c=2(2a+b)+c=4a+2b+c$

$a*(b*c)=a*(2b+c)=2a+2b+c$

Since $(a*b)*c\neq a*(b*c)$,

Therefore, binary operation * is not associative.

Question 9

X and Y are two points with position vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio 2 : 1 externally.

OR

Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-j+2\hat{k}$ be two vectors. Show that the vectors $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$ are perpendicular to each other.

Solution

The position vectors given are

$\overrightarrow{OX}=3\vec{a}+\vec{b}$ and $\overrightarrow{OY}=\vec{a}-3\vec{b}$

The position vector of the point Z which divides the line segment XY in the ratio 2 : 1 externally will be

$$\overrightarrow{OZ}=\frac{2\left(\vec{a}-3\vec{b}\right)-\left(3\vec{a}+\vec{b}\right)}{2-1}=\frac{-\vec{a}-7\vec{b}}{1}$$

$$\Rightarrow \overrightarrow{OZ}=-\vec{a}-7\vec{b}$$

OR

$$\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$$

$$\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$$

$$\vec{a}+\vec{b}=\left(\hat{i}+2\hat{j}-3\hat{k}\right)+\left(3\hat{i}-\hat{j}+2\hat{k}\right)$$

$$=\left(4\hat{i}+\hat{j}-\hat{k}\right)$$

$$\vec{a}-\vec{b}=\left(\hat{i}+2\hat{j}-3\hat{k}\right)-\left(3\hat{i}-\hat{j}+2\hat{k}\right)$$

$$=\left(-2\hat{i}+3\hat{j}-5\hat{k}\right)$$

$$\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=\left(4\hat{i}+\hat{j}-\hat{k}\right)\cdot\left(-2\hat{i}+3\hat{j}-5\hat{k}\right)$$

$$=-8+3+5$$

$$=0$$

Since the dot product of $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$ is 0 so, they are perpendicular to each other.

Question 10

Out of 8 outstanding students of a school, in which there are 3 boys and 5 girls, a team of 4 students is to be selected for a quiz competition. Find the probability that 2 boys and 2 girls are selected.

OR

In a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution

Total outstanding students = 8
Number of students to be selected = 4
Number of boys = 3
Number of girls = 5
Out of 8 students 4 students are to be selected in which 2 should be boys and 2 girls.
Ways of selecting 2 boys and 2 girls = ${}^3C_2 \times {}^5C_2$
P(Selecting 4 students are to be selected in which 2 should be boys and 2 girls) = $\frac{{}^3C_2 \times {}^5C_2}{{}^8C_4} = \frac{3}{7}$

OR

The repeated guessing of correct answers from multiple choice questions are Bernoulli trials. Let X represent the number of correct answers by guessing in the set of 5 multiple choice questions.

Probability of getting a correct answer is, $p = \frac{1}{3}$

$$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Clearly, X has a binomial distribution with $n = 5$ and $p = \frac{1}{3}$

$$\therefore P(X = x) = {}^nC_x q^{n-x} p^x$$
$$= {}^5C_x \left(\frac{2}{3}\right)^{5-x} \cdot \left(\frac{1}{3}\right)^x$$

P (guessing more than 4 correct answers) = P(X ≥ 4)

$$= P(X=4)+P(X=5)$$

$$= {}^5C_4\left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right)^4 + {}^5C_5\left(\frac{1}{3}\right)^5$$

$$= 5\cdot\frac{2}{3}\cdot\frac{1}{81}+1\cdot\frac{1}{243}$$

$$= \frac{10}{243}+\frac{1}{243}$$

$$= \frac{11}{243}$$

Question 11

The probabilities of solving a specific problem independently by A and B are 1/3 and 1/5 respectively. If both try to solve the problem independently, find the probability that the problem is solved.

SOLUTION:

Probability of solving the problem by A, P(A) = $\frac{1}{3}$

$$P\left(\overline{A}\right) = \frac{2}{3}$$

Probability of solving the problem by B, P(B) = $\frac{1}{5}$

$$P\left(\overline{B}\right) = \frac{4}{5}$$

Since the problem is solved independently by A and B,

Probability that the problem is solved = $P(A)\cdot P\left(\overline{B}\right)+P(B)\cdot P\left(\overline{A}\right)+P(A)\cdot P(B)$

$$= \frac{1}{3}\times\frac{4}{5}+\frac{1}{5}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{5}$$

$$= \frac{7}{15}$$

Question 12

For the matrix A = $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, find (A + A') and verify that it is a symmetric matrix.

Solution

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

$$A + A' = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 14 \end{bmatrix}$$

$$(A + A')' = \begin{bmatrix} 4 & 8 \\ 8 & 14 \end{bmatrix} = (A + A')$$

Thus, $(A + A')$ is a symmetric matrix.

Question 13

A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall.

SOLUTION

Let *y* m be the height of the wall at which the ladder touches. Also, let the foot of the ladder be *x* m away from the wall.

Then, by Pythagoras theorem, we have:

$x^2 + y^2 = 169$ [Length of the ladder = 13 m]

$$\Rightarrow y = \sqrt{169 - x^2}$$

Then, the rate of change of height (y) with respect to time (t) is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{169 - x^2}} \frac{dx}{dt}$$

It is given that $\frac{dx}{dt} = 2$ cm/s.

$$\frac{dy}{dt} = \frac{-2x}{\sqrt{169 - x^2}}$$

Now, when $x = 5$ m, we have:

$$\frac{dy}{dt} = \frac{-2 \times 5}{\sqrt{169 - 5^2}} = \frac{-10}{\sqrt{144}} = -\frac{5}{6}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{5}{6}$ cm/sec.

Question 14

Prove that :

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$$

Solution

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad ...(1)$

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12} \qquad ...(2)$

Let $\sin^{-1}\frac{56}{65}=z$. Then, $\sin z=\frac{56}{65}\Rightarrow \cos z=\frac{33}{65}$

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \sin^{-1}\frac{56}{65}=\tan^{-1}\frac{56}{33} \qquad ...(3)$

Now, we have:

$$
\begin{aligned}
\text{L.H.S.} &= \cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5} \\
&= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{3}{4} && \left[\text{Using (1) and (2)}\right] \\
&= \tan^{-1}\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\cdot\frac{3}{4}} && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right] \\
&= \tan^{-1}\frac{20+36}{48-15} \\
&= \tan^{-1}\frac{56}{33} \\
&= \sin^{-1}\frac{56}{65}=\text{R.H.S} && \left[\text{Using (3)}\right]
\end{aligned}
$$

Question 15

Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, and hence evaluate $\int_0^1 x^2 (1-x)^n\,dx$.

SOLUTION:

To prove: $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$

Proof: Let

$t = a - x$

$\Rightarrow dt = -dx$

When $x = 0,\ t = a$

When $x = a,\ t = 0$

Putting the value of x in LHS

$\int_a^0 f(a-t)(-\mathrm{d}t)$

$= -\int_a^0 f(a-t)(\mathrm{d}t)$

$= \int_0^a f(a-t)(\mathrm{d}t)$

$= \int_0^a f(a-x)(\mathrm{d}x) \qquad \left(\because \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x\right)$

$= \mathrm{RHS}$

Using this we can solve the given question as follows:

$\int_0^1 x^2(1-x)^n\,\mathrm{d}x$

$= \int_0^1 (1-x)^2(1-(1-x))^n\,\mathrm{d}x$

$= \int_0^1 \left(1+x^2-2x\right)(x)^n\,\mathrm{d}x$

$= \int_0^1 \left(x^n + x^{2+n} - 2x^{n+1}\right)dx$

$= \left[\frac{x^{n+1}}{n+1} + \frac{x^{n+3}}{n+3} - 2\frac{x^{n+2}}{n+2}\right]_0^1$

$= \left[\frac{1}{n+1} + \frac{1}{n+3} - \frac{2}{n+2}\right]$

Question 16

If $x = \sin t$, $y = \sin pt$, prove that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

OR

Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ with respect to $\cos^{-1}x^2$.

Solution

$x = \sin t$

$\Rightarrow \frac{dx}{dt} = \cos t$

$y = \sin pt$

$\Rightarrow \frac{dy}{dt} = p\cos pt$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t} = \frac{p\sqrt{1-\sin^2 pt}}{\sqrt{1-\sin^2 t}}$

$\Rightarrow \frac{dy}{dx} = \frac{p\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Squaring both sides we get

$\left(\frac{dy}{dx}\right)^2 = \frac{p^2(1-y^2)}{(1-x^2)}$

$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = p^2\left(1-y^2\right)$

Differentiating with respect to x

$\left(1-x^2\right)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}\left(1-x^2\right) = p^2\left(-2y\frac{dy}{dx}\right)$

$\Rightarrow \left(1-x^2\right)2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2(-2x) = -2yp^2\frac{dy}{dx}$

$\Rightarrow 2\left(\frac{dy}{dx}\right)\left\{\left(1-x^2\right)\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) + yp^2\right\} = 0$

Either $2\left(\frac{dy}{dx}\right) = 0$ or $\left\{\left(1-x^2\right)\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) + yp^2\right\} = 0$

But $2\left(\frac{dy}{dx}\right) \neq 0$

So, $\left\{\left(1-x^2\right)\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) + yp^2\right\} = 0$

OR

Let $u=\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ and $v=\cos^{-1}x^2$

Putting $x^2=\cos\theta$ in u

$u=\tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right]$

$\Rightarrow u=\tan^{-1}\left[\frac{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}\right]$

$\Rightarrow u=\tan^{-1}\left[\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right]$

$\Rightarrow u=\tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$

$\Rightarrow u=\tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$

$\Rightarrow u=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$

$\Rightarrow u=\frac{\pi}{4}-\frac{\theta}{2}$

$\Rightarrow u=\frac{\pi}{4}-\frac{\cos^{-1}x^2}{2}$

$\Rightarrow \frac{du}{dx}=0-\frac{1}{2}\left(\frac{-1}{\sqrt{1-x^4}}(2x)\right)$

$\Rightarrow \frac{du}{dx}=\frac{x}{\sqrt{1-x^4}}$

Also, $v=\cos^{-1}x^2$

$\frac{dv}{dx}=\frac{-1}{\sqrt{1-x^4}}(2x)=\frac{-2x}{\sqrt{1-x^4}}$

$\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$=\frac{\frac{x}{\sqrt{1-x^4}}}{\frac{-2x}{\sqrt{1-x^4}}}$

$=\frac{-1}{2}$

Question 17

Integrate the function $\frac{\cos(x+a)}{\sin(x+b)}$ w.r.t. x.

SOLUTION:

Let $I=\int \frac{\cos(x+a)}{\sin(x+b)}dx$. Then

$$I=\int \frac{\cos(x+b+a-b)}{\sin(x+b)}dx$$

$$I=\int \frac{\cos\{(x+b)+(a-b)\}}{\sin(x+b)}dx$$

$$I=\int \frac{\cos(x+b)\cdot\cos(a-b)-\sin(x+b)\cdot\sin(a-b)}{\sin(x+b)}dx$$

$$I=\int [\cos(a-b)\cdot\cot(x+b)-\sin(a-b)]dx$$

$$I=\cos(a-b)\cdot\log|\sin(x+b)|-x\cdot\sin(a-b)$$

Question 18

Let $A = R - (2)$ and $B = R - (1)$. If $f: A \to B$ is a function defined by $f(x)=\frac{x-1}{x-2}$, show that f is one-one and onto. Hence, find f^{-1}.

OR

Show that the relation S in the set $A = [x \in Z: 0 \le x \le 12]$ given by $S = [(a, b) : a, b \in Z, |a - b|$ is divisible by 3] is an equivalence relation.

SOLUTION:

A = **R** − {2}, B = **R** − {1}

f: A → B is defined as $f(x)=\frac{x-1}{x-2}$.

Let $x,\ y \in$ A such that $f(x)=f(y)$.

$$\Rightarrow \frac{x-1}{x-2}=\frac{y-1}{y-2}$$

$$\Rightarrow (x-1)(y-2)=(x-2)(y-1)$$

$$\Rightarrow xy-2x-y+2=xy-x-2y+2$$

$$\Rightarrow -2x-y=-x-2y$$

$$\Rightarrow 2x-x=2y-y$$

$$\Rightarrow x=y$$

∴ f is one-one.

Let $y \in B = \mathbf{R} - \{1\}$. Then, $y \neq 1$.

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now,

$$f(x) = y$$

$$\Rightarrow \frac{x-1}{x-2} = y$$

$$\Rightarrow x - 1 = y(x-2)$$

$$\Rightarrow x(1-y) = 1 - 2y$$

$$\Rightarrow x = \frac{1-2y}{1-y} \in \mathrm{A} \qquad [y \neq 1]$$

Thus, for any $y \in$ B, there exists $x = \frac{1-2y}{1-y} \in \mathrm{A}$ such that

$$f\left(\frac{1-2y}{1-y}\right) = \frac{\frac{1-2y}{1-y} - 1}{\frac{1-2y}{1-y} - 2} = \frac{1-2y-1+y}{1-2y-2+2y} = \frac{-y}{-1} = y$$

Therefore, f is onto.

Hence, function f is one-one and onto.

$$f^{-1}(x) = \frac{1-2x}{1-x}$$

OR

$$A = \{x \in \mathbf{Z} : 0 \leq x \leq 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

(i) $R = \{(a,b) : |a-b| \text{ is divisible by } 3\}$

For any element $a \in$ A, we have $(a, a) \in$ R as $|a-a| = 0$ is divisible by 3.

$\therefore$ R is reflexive.

Now, let $(a, b) \in R \Rightarrow |a-b|$ is divisible 3.

$\Rightarrow |-(a-b)| = |b-a|$ is divisible by 3

$\Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

Now, let $(a, b), (b, c) \in$ R.

$\Rightarrow |a-b|$ is divisible by 3 and $|b-c|$ is divisible by 3.
$\Rightarrow (a-b)$ is divisible by 3and $(b-c)$ is divisible by 3.
$\Rightarrow (a-c) = (a-b) + (b-c)$ is divisible by 3.
$\Rightarrow |a-c|$ is divisible by 3.
$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

Question 19

Solve the differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$ when $x = 0$.

OR

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$, given that $y = 1$ when $x = 0$

SOLUTION:

The given differential equation is:

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$$

$$\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$\Rightarrow \frac{dy}{1+y^2} = (1+x^2)dx$$

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1+y^2} = \int (1+x^2)dx$$

$$\Rightarrow \tan^{-1} y = \int dx + \int x^2 dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

It is given that $y = 1$ when $x = 0$.

$$\Rightarrow \tan^{-1}(1) = 0 + \frac{0^3}{3} + C$$

$$\Rightarrow C = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

This is the required solution of the given differential equation.

OR

The given differential equation is $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ .....(i)

Let $y = vx$, then .....(ii)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ .....(iii)

from (i), (ii) and (iii), we get

$v + x\frac{dv}{dx} = \frac{vx^2}{x^2+v^2x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2}{1+v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2-v-v^3}{1+v^2}$

$\Rightarrow \frac{(1+v^2)}{v^3+v-v^2}dv = -\frac{dx}{x}$

$\Rightarrow \frac{(v^2+1-v+v)}{v(v^2+1-v)}dv = -\frac{dx}{x}$

$\Rightarrow \left(\frac{1}{v} + \frac{1}{v^2+1-v}\right)dv = -\frac{dx}{x}$

Now, Integrate both the sides

$\Rightarrow \int\left(\frac{1}{v} + \frac{1}{v^2+1-v}\right)dv = -\int\frac{dx}{x}$

$\Rightarrow \int\frac{1}{v}dv + \int\frac{1}{v^2+1-v}dv = -\int\frac{dx}{x}$

$\Rightarrow \int\frac{1}{v}dv + \int\frac{1}{v^2-2\cdot v\cdot\frac{1}{2}+\frac{1}{4}+1-\frac{1}{4}}dv = -\int\frac{dx}{x}$

$\Rightarrow \int\frac{1}{v}dv + \int\frac{1}{\left(v-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv = -\int\frac{dx}{x}$

$\Rightarrow \ln v + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) = -\ln x + c$

$\Rightarrow \ln y + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y-x}{\sqrt{3}x}\right) = c$

It is given that $y = 1$ when $x = 0$,

Therefore $c = \frac{\pi}{\sqrt{3}}$

Hence, the particular solution of the given differential equation is $\ln y + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y-x}{\sqrt{3}x}\right) = \frac{\pi}{\sqrt{3}}$.

Question 20

Using properties of determinants, find the value of x for which

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0.$$

SOLUTION:

Given:

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$$R_1 \to R_1 + R_2 + R_3$$

$$\begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$$\text{Take } (12+x) \text{ common}$$

$$(12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$$

$$(12+x)\begin{vmatrix} 1 & 0 & 0 \\ 4+x & -2x & 0 \\ 4+x & 0 & -2x \end{vmatrix} = 0$$

$$(12+x)(-2x)(-2x) = 0$$

$$\Rightarrow x = 0, -12$$

Question 21

Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right), -4=0,\ \vec{r}.\ \left(2\hat{i}+\hat{j}-\hat{k}\right) + 5 = 0$ and which is perpendicular to the plane

$\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

SOLUTION:

The equations of the given planes are

$$\vec{r}\cdot\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0 \qquad ...(1)$$
$$\vec{r}\cdot\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0 \qquad ...(2)$$

The equation of the plane passing through the line intersection of the plane given in equation (1) and equation (2) is

$$\left[\vec{r}\cdot\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}\cdot\left(2\hat{i}+\hat{j}-\hat{k}\right)+5\right]=0$$
$$\vec{r}\cdot\left[(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\right]+(5\lambda-4)=0 \qquad ...(3)$$

The plane in equation (3) is perpendicular to the plane, $\vec{r}\cdot\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

$$\therefore 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$$
$$\Rightarrow 19\lambda-7=0$$
$$\Rightarrow \lambda=\frac{7}{19}$$

Substituting $\lambda=\frac{7}{19}$ in equation (3), we obtain

$$\Rightarrow \vec{r}\cdot\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right]\frac{-41}{19}=0$$
$$\Rightarrow \vec{r}\cdot\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0 \qquad ...(4)$$

This is the vector equation of the required plane.

Question 22

Find the value of *x* such that the four point with position vectors,
$A\left(3\hat{i}+2\hat{j}+\hat{k}\right)$, $B\left(4\hat{i}+x\hat{j}+5\hat{k}\right)$, $C\left(4\hat{i}+2\hat{j}-2\hat{k}\right)$ and $D\left(6\hat{i}+5\hat{j}-\hat{k}\right)$ are coplanar.

SOLUTION:

Let A, B, C, D be the given points. Then,

$$\overrightarrow{AB}=\left(4\hat{i}+x\hat{j}+5\hat{k}\right)-\left(3\hat{i}+2\hat{j}+\hat{k}\right)=\hat{i}+(x-2)\hat{j}+4\hat{k}$$

$$\overrightarrow{AC}=\left(4\hat{i}+2\hat{j}-2\hat{k}\right)-\left(3\hat{i}+2\hat{j}+\hat{k}\right)=\hat{i}+0\hat{j}-3\hat{k}$$

$$\overrightarrow{AD}=\left(6\hat{i}+5\hat{j}-\hat{k}\right)-\left(3\hat{i}+2\hat{j}+\hat{k}\right)=3\hat{i}+3\hat{j}-2\hat{k}$$

The given points are coplanar iff vectors $\overrightarrow{AB},\ \overrightarrow{AC},\ \overrightarrow{AD}$ are coplanar.
Therefore,

$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0$$

$$\Rightarrow\begin{vmatrix}1 & (x-2) & 4\\ 1 & 0 & -3\\ 3 & 3 & -2\end{vmatrix}=0$$

$$\Rightarrow 1\left(0+9\right)-(x-2)\left(-2+9\right)+4\left(3-0\right)=0$$

$$\Rightarrow 35-7x=0$$

$$\Rightarrow x=5$$

Hence, all the four points are coplanar for $x=5$.

Question 23

If $y=(\log x)^x+x^{\log x}$, find $\frac{dy}{dx}$.

SOLUTION:

Let $y=\left(\log x\right)^x+x^{\log x}$

Also, let $u=\left(\log x\right)^x$ and $v=x^{\log x}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\text{...(1)}$$

$u=(\log x)^x$

$$\Rightarrow\log u=\log\left[\left(\log x\right)^x\right]$$

$$\Rightarrow\log u=x\log\left(\log x\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad ...(2)$$

$$v=x^{\log x}$$

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^{2}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}\cdot\log x\qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}\cdot\log x$$

Question 24

Find the vector equation of a line passing through the point (2, 3, 2) and parallel to the line $\vec{r} = \left(-2\hat{i} + 3\hat{j}\right) + \lambda\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right)$. Also, find the distance between these two lines.

OR

Find the coordinates of the foot of the perpendicular *Q* drawn from *P*(3, 2, 1) to the plane $2x - y + z + 1 = 0$. Also, find the distance *PQ* and the image of the point *P* treating this plane as a mirror

SOLUTION:

It is given that line passes through the point (2,3,2) and is parallel to the line $\vec{r} = \left(-2\hat{i} + 3\hat{j}\right) + \lambda\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right)$.

i.e. required line is parallel to the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Equation of the required line is $\vec{r} = \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) + \lambda\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right)$

The two lines are parallel, we have

$\vec{a_1} = -2\hat{i} + 3\hat{j}$, $\vec{a_2} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Therefore, the distance between the lines is given by

$$d = \left|\frac{\vec{b} \times \left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b}\right|}\right| = \left|\frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 6 \\ 4 & 0 & 2 \end{vmatrix}}{\sqrt{4+9+36}}\right|$$

$$= \frac{\left|-6\hat{i} + 20\hat{j} + 12\hat{k}\right|}{\sqrt{49}} = \frac{\sqrt{580}}{\sqrt{49}} = \frac{2\sqrt{145}}{7}$$

OR

The given equation of the plane is $2x - y + z + 1 = 0$.
Foot of the perpendicular is given by
$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = -\frac{(ax_1+by_1+cz_1-d)}{a^2+b^2+c^2}$
Therefore, foot of perpendicular Q from point P(3,2,1) to the given plane is given by
$\frac{x-3}{2} = \frac{y-2}{-1} = \frac{z-1}{1} = -\frac{(2\times3+(-1)\times2+1\times1+1)}{2^2+(-1)^2+1^2} = -1$

$\Rightarrow x = 1,\ y = 3,\ z = 0$
Hence, coordinates of foot of perpendicular Q are $(1, 3, 0)$.

Distance PQ=$\left|\frac{(2\times3+(-1)\times2+1\times1+1)}{\sqrt{2^2+(-1)^2+1^2}}\right| = \sqrt{6}$

Image of the point P is given by
$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = -\frac{2(ax_1+by_1+cz_1-d)}{a^2+b^2+c^2}$
Therefore, coordinates of image of the point P are given by
$\frac{x-3}{2} = \frac{y-2}{-1} = \frac{z-1}{1} = -\frac{2[(2\times3+(-1)\times2+1\times1+1)]}{2^2+(-1)^2+1^2} = -2$

$\Rightarrow x = -1,\ y = 4,\ z = -1$
Hence, coordinates of image of P are $(-1, 4, -1)$.

Question 25

Using elementary row transformation, find the inverse of the matrix

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

OR

Using matrices, solve the following system of linear equations :

$x + 2y - 3z = -4$
$2x + 3y + 2z = 2$
$3x - 3y - 4z = 11$

SOLUTION:

We have $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$A = IA$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & -2 \\ 3 & 2 & -4 \\ 2 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_1,$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & -5 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_1 \to R_1 + R_2,$

$R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & -3 \\ 1 & -5 & 13 \end{bmatrix} A$$

$R_2 \to -R_2,$

$R_3 \to -R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & 3 \\ -1 & 5 & -13 \end{bmatrix} A$$

$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} A$$

We know, $I = AA^{-1}$

Therefore, inverse of A i.e. $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

OR

The system of equations can be written in the form $AX = B$, where

$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

$|A| = 1(-12+6) - 2(-8-6) - 3(-6-9) = 67 \neq 0$

Therefore, A is non singular and so its inverse exists.

$A_{11} = -6,\ A_{12} = 14,\ A_{13} = -15$

$A_{21} = 17,\ A_{22} = 5,\ A_{23} = 9$

$A_{31} = 13,\ A_{32} = -8,\ A_{33} = -1$

Therefore, $A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$

So $X = A^{-1}B = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Hence, $x = 3,\ y = -2$ and $z = 1$

Question 26

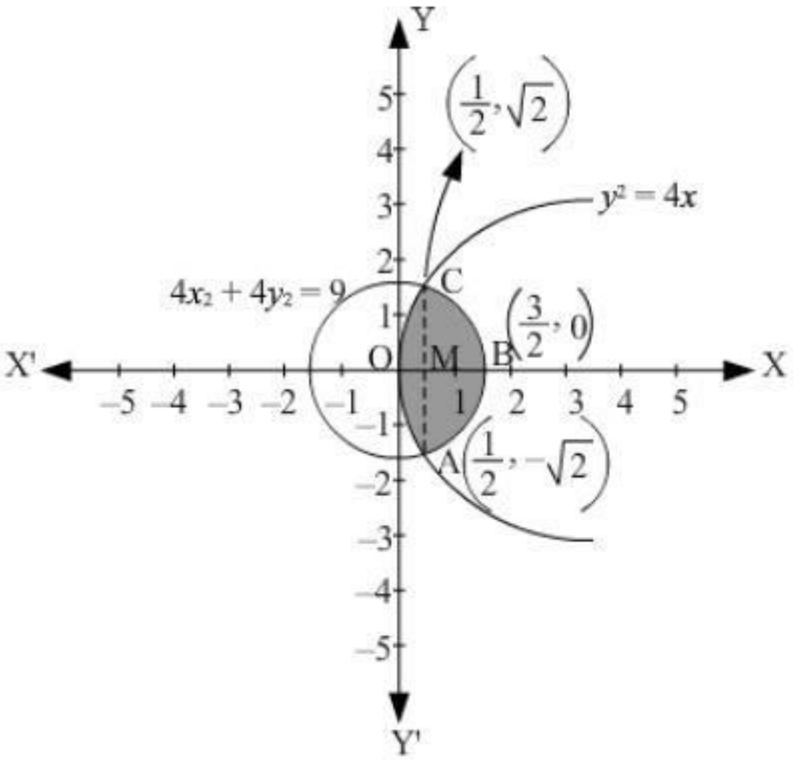
Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.

OR

Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$

SOLUTION:

The area bounded by the parabola $y^2 = 4x$ and circle $4x^2 + 4y^2 = 9$, is represented as



The points of intersection of both the curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The required area is given by OABCO.

It can be observed that area OABCO is symmetrical about *x*-axis.

∴ Area OABCO = 2 × Area OBC

Area OBCO = Area OMC + Area MBC

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9 - 4x^2}\, dx$$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{(3)^2 - (2x)^2}\, dx$$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

When $x = \frac{3}{2}, t = 3$ and when $x = \frac{1}{2}, t = 1$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \frac{1}{4}\int_1^3 \sqrt{(3)^2 - (t)^2}\, dt$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}} + \frac{1}{4}\left[\frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_1^3$$

$$= 2\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right] + \frac{1}{4}\left[\left\{\frac{3}{2}\sqrt{9-(3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right\} - \left\{\frac{1}{2}\sqrt{9-(1)^2} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{2}{3\sqrt{2}} + \frac{1}{4}\left[\left\{0 + \frac{9}{2}\sin^{-1}(1)\right\} - \left\{\frac{1}{2}\sqrt{8} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{\sqrt{2}}{3} + \frac{1}{4}\left[\frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{3} + \frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}$$

Therefore, the required area is $\left[2\times\left(\frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}\right)\right] = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}$ sq.units

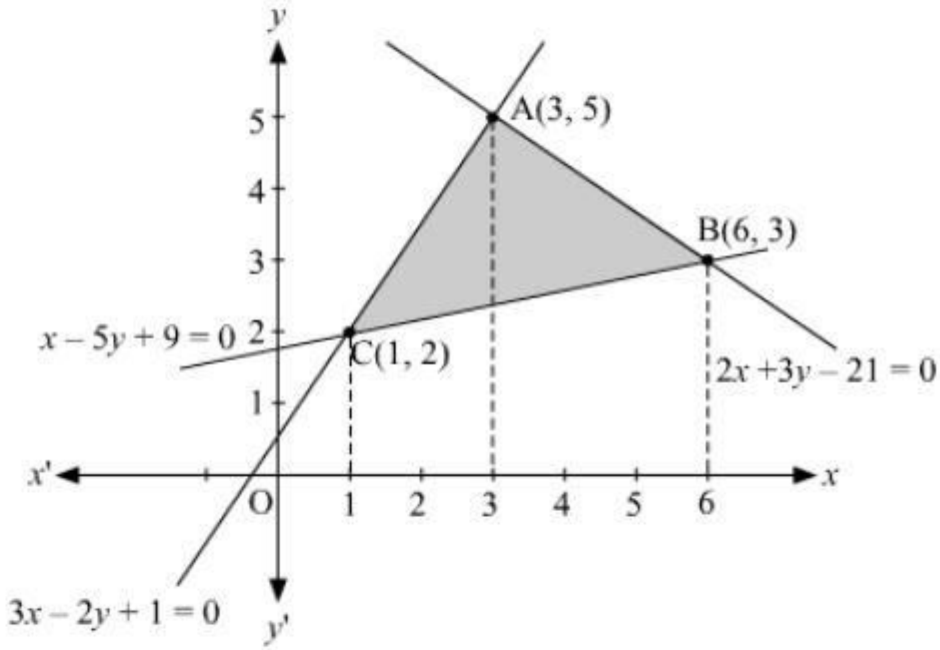
OR

$3x - 2y + 1 = 0 \Rightarrow y_1 = \frac{(3x+1)}{2}$ .....(i)

$2x + 3y - 21 = 0 \Rightarrow y_2 = \frac{(21-2x)}{3}$ .....(ii)

$x - 5y + 9 = 0 \Rightarrow y_3 = \frac{(x+9)}{5}$ .....(iii)

Point of intersection of (i) and (ii) is A(3, 5)
Point of intersection of (ii) and (iii) is B(6, 3) and
Point of intersection of (iii) and (i) is C(1, 2).



Therefore, area of the region bounded $=\int_1^3 y_1 \cdot dx+\int_3^6 y_2 \cdot dx-\int_1^6 y_3 \cdot dx$

$$=\int_1^3 \frac{(3x+1)}{2} \cdot dx+\int_3^6 \frac{(21-2x)}{3} \cdot dx-\int_1^6 \frac{(x+9)}{5} \cdot dx$$

$$=\frac{1}{2}\left(\frac{3x^2}{2}+x\right)_1^3+\frac{1}{3}\left(21x-x^2\right)_3^6-\frac{1}{5}\left(\frac{x^2}{2}+9x\right)_1^6$$

$$=\frac{1}{2}[14]+\frac{1}{3}[36]-\frac{1}{5}\left[\frac{125}{2}\right]$$

$$=7+12-12.5$$

$$=6.5 \text{ sq. units}$$

Question 27

An insurance company insured 3000 cyclists, 6000 scooter drivers and 9000 car drivers. The probability of an accident involving a cyclist, a scooter driver and a car driver are 0·3, 0·05 and 0·02 respectively. One of the insured persons meets with an accident. What is the probability that he is a cyclist?

SOLUTION:

Let E_1, E_2, and E_3 be the respective events that the driver is a cyclist, a scooter driver, and a car driver.

Let A be the event that the person meets with an accident.

There are 3000 cyclists, 6000 scooter drivers, and 9000 car drivers.

Total number of drivers = 3000 + 6000 + 9000 = 18000

$P(E_1)$ = P(driver is a cyclist) $= \frac{3000}{18000} = \frac{1}{6}$

$P(E_2)$ = P(driver is a scooter driver) $= \frac{6000}{18000} = \frac{1}{3}$

$P(E_3)$ = P(driver is a car driver)$= \frac{9000}{18000} = \frac{1}{2}$

$$\mathrm{P}(\mathrm{A}|\mathrm{E}_1) = \mathrm{P}(\text{cyclist met with an accident}) = 0.3$$
$$\mathrm{P}(\mathrm{A}|\mathrm{E}_2) = \mathrm{P}(\text{scooter driver met with an accident}) = 0.05$$
$$\mathrm{P}(\mathrm{A}|\mathrm{E}_3) = \mathrm{P}(\text{car driver met with an accident}) = 0.02$$

The probability that the driver is a cyclist, given that he met with an accident, is given by $P(E_1|A)$.

$$\mathrm{P}(\mathrm{E}_1|\mathrm{A}) = \frac{\mathrm{P}(\mathrm{E}_1)\cdot\mathrm{P}(\mathrm{A}|\mathrm{E}_1)}{\mathrm{P}(\mathrm{E}_1)\cdot\mathrm{P}(\mathrm{A}|\mathrm{E}_1)+\mathrm{P}(\mathrm{E}_2)\cdot\mathrm{P}(\mathrm{A}|\mathrm{E}_2)+\mathrm{P}(\mathrm{E}_3)\cdot\mathrm{P}(\mathrm{A}|\mathrm{E}_3)}$$
$$= \frac{\frac{1}{6}\times 0.3}{\frac{1}{6}\times 0.3+\frac{1}{3}\times 0.05+\frac{1}{2}\times 0.02}$$
$$= \frac{\frac{1}{6}\times\frac{30}{100}}{\frac{1}{6}\times\frac{30}{100}+\frac{2}{6}\times\frac{5}{100}+\frac{3}{6}\times\frac{2}{100}}$$
$$= \frac{30}{46} = \frac{15}{23}$$

Question 28

Using integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

SOLUTION:

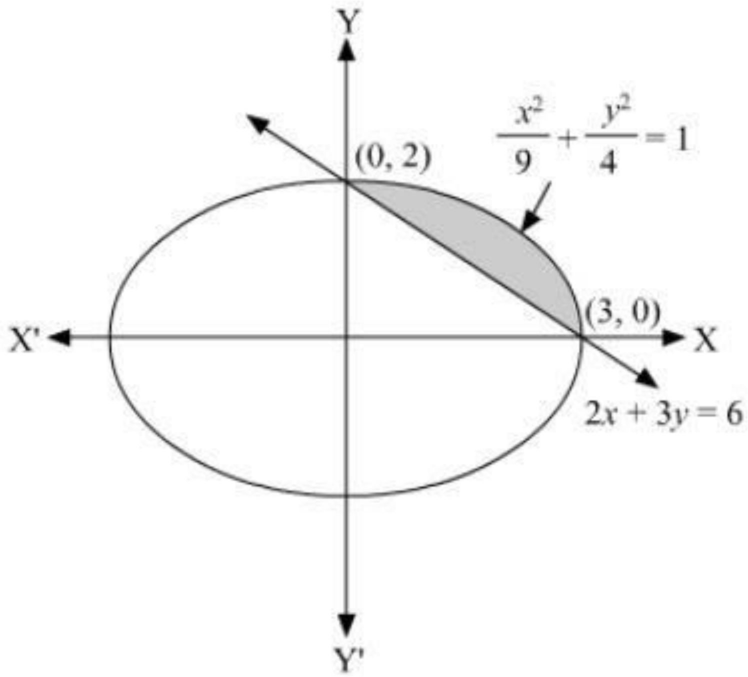
Given equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$

Let $y_1 = \frac{2}{3}\sqrt{9-x^2}$ and

equation of the line is $\frac{x}{3}+\frac{y}{2}=1$

Let $y_2 = \frac{2}{3}(3-x)$

we have $(3,0)$ and $(0,2)$ as the points of intersection of ellipse and line.



Therefore, area of smaller region,A$=\int_0^3 (y_1-y_2)dx$

$$A=\int_0^3\left[\frac{2}{3}\sqrt{9-x^2}-\frac{2}{3}(3-x)\right]dx$$

$$=\int_0^3\left(\frac{2}{3}\sqrt{9-x^2}\right)dx-\int_0^3\left[\frac{2}{3}(3-x)\right]dx$$

$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3-\frac{2}{3}\left(3x-\frac{x^2}{2}\right)_0^3$$

$$=\frac{2}{3}\left[\left(0+\frac{9}{2}\times\frac{\pi}{2}\right)-0\right]-\frac{2}{3}\left(9-\frac{9}{2}-0\right)$$

$$=\left(\frac{3\pi}{2}-3\right)\text{ sq. unit}$$

Question 29

A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 35 per package of nuts and ₹ 14 per package of bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates each machine for atmost 12 hours a day? Convert it into an LPP and solve graphically.

SOLUTION:

Let the manufacturer produce x packages of nuts and y packages of bolts. Therefore,

$x \geq 0$ and $y \geq 0$

The given information can be compiled in a table as follows.

	Nuts	Bolts	Availability
Machine A (h)	1	3	12
Machine B (h)	3	1	12

The profit on a package of nuts is Rs 35 and on a package of bolts is Rs 14. Therefore, the constraints are

$x + 3y \le 12$

$3x + y \le 12$

Total profit, $Z = 35x + 14y$

The mathematical formulation of the given problem is

Maximise $Z = 35x + 14y$ ... (1)

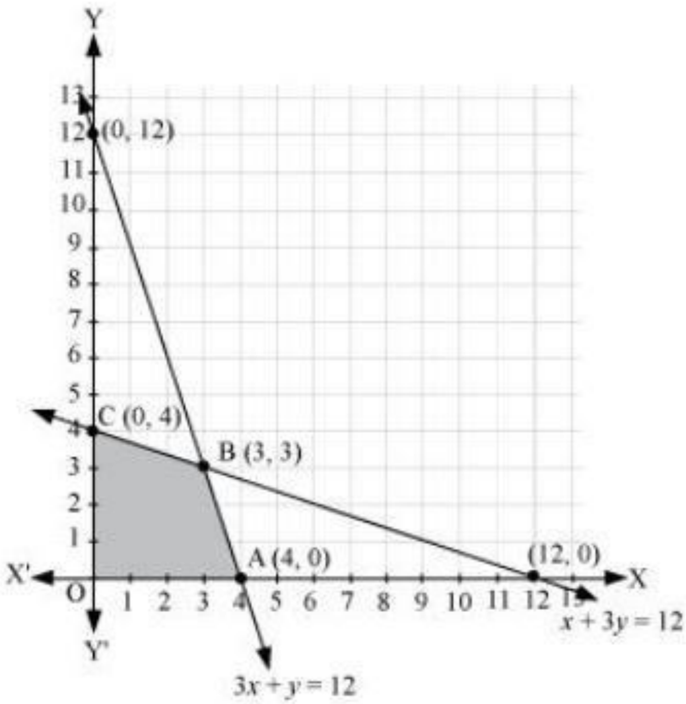
subject to the constraints,

$x + 3y \le 12$ ... (2)

$3x + y \le 12$ ... (3)

$x, y \ge 0$ ... (4)

The feasible region determined by the system of constraints is as follows.



The corner points are A(4, 0), B(3, 3), and C(0, 4).

The values of Z at these corner points are as follows.

Corner point	$Z = 35x + 14y$	
O(0, 0)	0	
A(4, 0)	140	
B(3, 3)	147	→ Maximum
C(0, 4)	56	

The maximum value of Z is Rs 147 at (3, 3).

Thus, 3 packages of nuts and 3 packages of bolts should be produced each day to get the maximum profit of Rs 147.